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Fibonacci orbits and $SU(2)$-dynamics

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Abstract. Motivated by an interpretation of the Fibonacci sequence as a subset of the free group F_2 we describe orbits on finite subgroups of $SU(2)$. A prerequisite for the calculation of these orbits is the knowledge of all generating sets of finite subgroups of $SU(2)$. In order to facilitate this problem we introduce an equivalence relation between generating sets which can be extended to another one between Fibonacci orbits. Additionally, we describe a possible experiment with quasiperiodic dynamics.

1. Introduction

In this paper, which is based on the diploma thesis [14], we develop a group-theoretical method to describe one-dimensional, quasiperiodic dynamics, where the dynamics is defined by a recursive mapping of the elements of $SU(2)$. Hence, one can consider the dynamics from two different points of view. For the first instance, disregarding the physical applications, the recursive mapping can be investigated as a dynamical system. As a consequence, we concentrate on the investigation of cases in which the recurrence relation results in periodic orbits.

The other point of view is the description of physical systems with quasiperiodic structure. In this paper, we consider quasiperiodic sequences of matrices of $SU(2)$. In so doing, we interpret the matrices of $SU(2)$ as time evolution operators of a two-level system, i.e. the underlying quasiperiodic structure is considered as a quasiperiodic sequence of time intervals.

The one-dimensional, quasiperiodic structure to be investigated is the Fibonacci sequence (resp. generalized Fibonacci sequences). The starting point of the algebraic approach is the fact that the local inflation rule of the Fibonacci sequence can be expressed as an element of the group of automorphisms of the free group F_2. If F_2 is given by $F_2 = \langle a, b\rangle$ then the Fibonacci sequence can be interpreted as the orbit starting at a under the positive powers of the automorphism. By homomorphisms this subset of F_2 can be mapped in each group which can be generated by two elements. The images of the Fibonacci sequence under the homomorphisms are called Fibonacci orbits. In sections 2 and 3 this is outlined in detail.

Section 4 deals with the determination and classification of all possibilities of generating a finite subgroup of $SU(2)$ by two elements. This analysis results in a representative of each equivalence class of generating sets. These representatives can be used to calculate the Fibonacci orbits by a recurrence relation, where the equivalence relation between generating sets can be extended in a natural manner to an equivalence relation between Fibonacci orbits. The definition of the equivalence relation together with other general remarks about

Fibonacci orbits can be found in section 5.1. In section 5.2, we show that the condition for the periodicity of a Fibonacci orbit on the cyclic subgroup C_n of $SU(2)$ can be formulated as an eigenvalue problem in the commutative ring $\mathbb{Z}/n\mathbb{Z}$. Actually, we cannot give the general solution of this problem but we can derive criteria which reduce the number of possible solutions

In section 6, we turn to the physical properties of a quasiperiodic system. The starting point is an interpretation of the well known spin echo experiment as a periodic $SU(2)$-dynamics. This interpretation bases on the fact that in the spin echo experiment a two-level system which passes alternatingly through two types of time intervals. As a consequence, one can measure a periodic $SO(3)$-dynamics of the expectations of the magnetic moment. By concatenating the two types of intervals of the spin echo experiment by the method of the Fibonacci sequence, one obtains a quasiperiodic version of the spin echo experiment.

2. The free group F_2 and Fibonacci automorphisms

In order to explain which maps we are interested in, let us start from the definition of free groups.

Definition 2.1. A group F_n is called free of rank n with generating set $\{f_i; 1 \leqslant i \leqslant n\}$, if the following statements are true:

(i) $F_n := \langle f_i; 1 \leqslant i \leqslant n\rangle$, i.e. F_n is generated from the f_i's.
(ii) If G is a group generated by n elements g_i $(1 \leqslant i \leqslant n)$, i.e. $G = \langle g_i; 1 \leqslant i \leqslant n \rangle$, then there exists an epimorphism (surjective homomorphism) $\mu : F_n \longmapsto G$, $\mu(f_i) = g_i$ for all $1 \leqslant i \leqslant n$.

Starting from this definition one can prove [4, chapter I, section 19], that free groups exist and are unique (up to isomorphisms) for all $n \in \mathbb{N}$. The existence proof in [4] contains the connection to another possibility of defining free groups (cf [9, p 12]): given the set $\Sigma := \{f_i, f_i^{-1}; 1 \leqslant i \leqslant n\}$. Any finite sequence formed from elements of Σ is called a word. Let the multiplication of words be defined by concatenation. If one introduces the equivalence relations $f_i^{-1} f_i = f_i f_i^{-1} = id$ (empty word), the set of all equivalence classes, usually represented by the so-called freely reduced words, becomes the free group F_n.

Let $G = \langle g_i; 1 \leqslant i \leqslant n \rangle$ be a group. Then there exists (compare definition 2.1) an epimorphism $\mu : F_n \longmapsto G$, $\mu(f_i) = g_i$ $(1 \leqslant i \leqslant n)$. According to the homomorphism theorem the following assertion is valid:

$$G \cong F_n / \mathrm{Ker}(\mu) \,. \tag{2.1}$$

Given a subset $S = \{s_1, \ldots, s_m\}$ $(m \in \mathbb{N} \cup \{\infty\})$ of $\mathrm{Ker}(\mu)$ with the property $\mathrm{Ker}(\mu) = \langle f s_1 f^{-1}, \ldots, f s_m f^{-1}; f \in F_n \rangle$, then the group $G \cong F_n / \mathrm{Ker}(\mu)$ is fixed uniquely (up to isomorphisms) by the generators f_i of F_n and the set S. This leads to the notation

$$G := \langle g_1, \ldots, g_n; r_1 = \ldots = r_m = id \rangle$$

where the definitions $g_i := \mu(f_i)$, $r_j := \mu(s_j)$ are used.

Consider the homomorphisms $\rho_\ell : F_2 = \langle a, b\rangle \longmapsto F_2 = \langle a, b\rangle$ $(\ell \in \mathbb{Z})$, which are defined by

$$\rho_\ell \begin{cases} a \mapsto b \\ b \mapsto b^\ell a \,. \end{cases} \tag{2.2}$$

These mappings have inverses

$$\rho_\ell^{-1} \begin{cases} a \mapsto a^{-\ell} b \\ b \mapsto a \,. \end{cases} \tag{2.3}$$

Thus, the ρ_ℓ's are elements of the group of automorphisms Φ_2 of the free group F_2. By induction one can show the identity

$$K(\rho_\ell^q(b), \rho_\ell^q(a)) = K(b, a)^{(-1)^q} \tag{2.4}$$

where the group commutator $K(g, h)$ is defined by $K(g, h) := g^{-1}h^{-1}gh$. Note, that equation (2.4) is a special case of Nielsen's result ([10]) that one can write $K(\xi(b), \xi(a))$ as

$$K(\xi(b), \xi(a)) = f^{-1} K(b, a)^{(-1)^m} f \tag{2.5}$$

$(f \in F_2, m \in \{1, 2\})$ for each $\xi \in \Phi_2$.

On the other hand, ρ_1 is the standard local inflation rule of the Fibonacci sequence. This allows an interpretation of the Fibonacci sequence in terms of F_2: The Fibonacci sequence is the orbit starting at a under the positive powers of the automorphism ρ_1, i.e.

$$\text{Fibonacci sequence} := \{\rho_1^n(a); n \in \mathbb{N}_0\} \tag{2.6}$$

where the multiplication of automorphisms $\rho\,\eta$ is defined by the concatenation of mappings $\eta \circ \rho$. Because of this interpretation we call the ρ_ℓ's (generalized) Fibonacci automorphisms (cf [2]).

3. Fibonacci sequences under homomorphisms

Given an epimorphism $\mu : F_2 = \langle a, b\rangle \longmapsto G = \langle x, y\rangle$; $\mu(a) = x$, $\mu(b) = y$. Consider the Fibonacci sequence $\{\rho_\ell^n(a); n \in \mathbb{N}_0\}$. The image of the Fibonacci sequence relating to the homomorphism μ is called Fibonacci orbit, i.e.

$$\text{Fibonacci orbit} := \{M_n;\ n \in \mathbb{N}_0\} := \{\mu(\rho_\ell^n(a));\ n \in \mathbb{N}_0\}\,. \tag{3.1}$$

By induction one can prove the recurrence relation

$$M_{n+1} = M_n^\ell\, M_{n-1} \tag{3.2}$$

which is equivalent to

$$M_{n-1} = M_n^{-\ell}\, M_{n+1}\,. \tag{3.3}$$

If there is defined an addition of group elements (e.g. if G is a matrix group), then (2.4) together with the homomorphism property of μ results in an invariant I of the Fibonacci orbit

$$I := K(M_1, M_0) + K(M_0, M_1)\,. \tag{3.4}$$

In this paper we investigate the cases in which $M_0, M_1 \in SU(2, \mathbb{C})$. Then, as a consequence of the Cayley–Hamilton theorem (cf [8, p 400]), we obtain the identity

$$I := \mathrm{tr}(K(M_0, M_1))\,\mathbf{1}\,. \tag{3.5}$$

Thus, if $M_0, M_1 \in SU(2, \mathbb{C})$, the invariant I is equivalent to the Fricke–Vogt invariant $\hat{I} = \frac{1}{4}\,\mathrm{tr}(K(M_0, M_1)) - \frac{1}{2}$ (cf [11]).

Definition 3.1. The Fibonacci orbit $\{M_n; n \in \mathbb{N}_0\}$ is called periodic, if there exists $p \in \mathbb{N}$ so that for all $m \in \mathbb{N}$, $j \in \{0, 1, \ldots, p-1\}$ the following identity is valid: $M_j = M_{j+mp}$. The smallest p with this property is called the period.

This definition of periodic Fibonacci orbits is reasonable, because a Fibonacci orbit cannot become periodic after finitely many steps. It is easy to check this assertion: assume that a Fibonacci orbit becomes periodic after finitely many steps, i.e. there exists a minimum $n \in \mathbb{N}$, so that $M_j = M_{j+mp}$ for all $n \leqslant j$. Then, because of (3.3), $M_{n-1} = M_{n-1+mp}$ which contradicts the assumption that n is the minimum number with this property.

Because iteration (3.2) has recursion depth 2, we can replace definition 3.1 by the equivalent definition 3.2, which is useful for the determination of periods:

Definition 3.2. The period p is the minimum natural number for which $M_0 = M_p$ and $M_1 = M_{p+1}$.

If the period is odd, definition 3.2 together with (2.4) results in a restriction of the group commutator: $K(M_0, M_1)$ must be the identity or an involution. Thus, if $M_0, M_1 \in SU(2)$, then $K(M_0, M_1) = \pm \mathbf{1}_2$. For consequences we refer to section 5.

The least common multiple of the orders of all elements of the group G is called the exponential $\exp(G)$ of G. Let a group G have a finite exponential and let ℓ, k be integers with the property $l = z \exp(G) + k$ $(z \in \mathbb{Z})$. Then $\mu(\rho_k^n(a)) = \mu(\rho_\ell^n(a))$ for all $n \in \mathbb{N}_0$, i.e. the Fibonacci orbits are the same.

In the following sections we consider the cases in which G is a finite subgroup of $SU(2)$. For these groups there always exists a finite exponential. Additionally all Fibonacci orbits are periodic, because there are at most $\mathrm{Ord}(G)^2$ ($\mathrm{Ord}(G)$:= order of G) different pairs (M_j, M_{j+1}) in a Fibonacci orbit.

Note that an investigation of all finite subgroups of $SU(2)$ can be regarded as an investigation of all finite subgroups of $SL(2, \mathbb{C})$, since each representation of a finite group can be taken (cf [10, p 66]) to a unitary representation by similarity transformation.

4. The generators of finite subgroups of $SU(2)$

4.1. Equivalence classes of generating sets

Let $F_2 = \langle a, b \rangle$ be the free group of rank 2. In order to determine all Fibonacci orbits in finite subgroups of $SU(2)$, we have to know all possibilities of choosing $\mu(a), \mu(b)$, so that $\langle \mu(a), \mu(b) \rangle$ is a finite subgroup of $SU(2)$. To facilitate this problem, we introduce the following equivalence relations:

Definition 4.1. The generating sets $E_1 = \{f_1, f_2\}$ and $E_2 = \{g_1, g_2\}$ are called $SU(2)$- resp. $SO(3)$-equivalent, if there exists $h \in SU(2)$ resp. $SO(3)$, so that $h^{-1} g_i h = f_i$ $(i = 1, 2)$.

Remark. Subsequently, if we talk about generating sets, we always mean generating sets consisting of two elements. The existence of such generating sets will be shown in section 4.2 (resp. 4.3).

We have chosen these equivalence relations because the conjugations with elements of $SU(2)$ resp. $SO(3)$ is the greatest subset of isomorphisms which preserves the essential properties of the representation matrices like unitarity resp. orthogonality, trace, and determinant. Additionally, conjugation with unitary operators is the equivalence relation which is used in quantum mechanics and the theory of Hilbert spaces.

The group $SU(2)$ can be characterized in the following way ($\sigma_0 := \mathbf{1}_2$; $\sigma_1, \sigma_2, \sigma_3$ Pauli matrices):

$$SU(2, \mathbb{C}) := \left\{ a_0 \sigma_0 + i \left(\sum_{j=1}^{3} a_j \sigma_j \right) ;\ a_j \in \mathbb{R}, \sum_{j=0}^{3} a_j^2 = 1 \right\}.$$

Let V be a vector space over the field $\mathbb{R}$ with basis $\{\sigma_1, \sigma_2, \sigma_3\}$. Then there exists a map $\Pi : SU(2) \longmapsto SO(3)$ defined by $\Pi(q)(p) := q^{-1}pq$ ($q \in SU(2)$, $p \in V$). Π is a two-to-one epimorphism ($\Pi(q) = \Pi(-q)$) onto $SO(3)$, i.e. $SU(2)$ is a double cover of $SO(3)$.

Definition 4.2. Let S be a subset of $SO(3)$; then $\Pi^{-1}(S)$ is defined by $\Pi^{-1}(S) := \{h \in SU(2), \Pi(h) \in S\}$.

The proof of the next theorem can be found in [16, p 83]:

Theorem 4.1. Every finite subgroup of $SO(3)$ is a cyclic group C_n, a dihedral group D_n, a tetrahedral group T, an octahedral group O, or an icosahedral group Y. Every finite subgroup of $SU(2)$ is a cyclic group C_n, a binary dihedral group $D_n^* = \Pi^{-1}(D_n)$, a binary tetrahedral group $T^* := \Pi^{-1}(T)$, a binary octahedral group $O^* := \Pi^{-1}(O)$, or a binary icosahedral group $Y^* := \Pi^{-1}(Y)$. If two finite subgroups of $SO(3)$ resp. $SU(2)$ are isomorphic, then they are conjugate in $SO(3)$ resp. $SU(2)$.

It is worth noting that the finite subgroups of $SU(2)$ can be associated with Dynkin diagrams (cf [13, appendix]).

Theorem 4.1 determines all isomorphism classes of finite subgroups of $SU(2)$. Another interesting assertion of the theorem is that all finite subgroups of $SO(3)$ resp. $SU(2)$ which are elements of the same isomorphism class are conjugate in $SO(3)$ resp. $SU(2)$. Hence, to determine all generating sets of an isomorphism class up to $SU(2)$-equivalence, it is sufficient to investigate the generating sets of a faithful representation.

Theorem 4.2. Let $G = \langle g_1, g_2 \rangle$ be a noncyclic finite subgroup of $SO(3)$. Given any $f_1 \in \Pi^{-1}(\{g_1\})$, $f_2 \in \Pi^{-1}(\{g_2\})$, then $F := \Pi^{-1}(G) = \langle f_1, f_2 \rangle$.

Proof. Π: $F \longmapsto G$ is an epimorphism with $\mathrm{Ker}(\Pi) = \{\mathbf{1}_2, -\mathbf{1}_2\} =: Z$. Thus, for each element $g \in G$ there exists precisely one coset Zt of Z ($t \in F$) with $Zt = \Pi^{-1}(\{g\})$. Because $G = \langle g_1, g_2 \rangle$, each $g \in G$ can be written in the form $g = g_{i_1} \dots g_{i_k}$ ($i_j \in \{1, 2\}, k \in \mathbb{N}$). Let f be the element of $\langle f_1, f_2 \rangle$ defined by $f := f_{i_1} \dots f_{i_k}$. Then $\Pi(f) = g$. Therefore, a representative of each coset of Z in F is contained in $\langle f_1, f_2 \rangle$. It remains to show that $-\mathbf{1}_2 \in \langle f_1, f_2 \rangle$.

Each finite, noncyclic subgroup of $SO(3)$ contains an involution. Let $g \in G$, $g^2 = \mathbf{1}_3$. As we have mentioned above, $\langle f_1, f_2 \rangle$ includes an element f with $\Pi(f) = g$. Because g is an involution, the order of f is 4. Since $-\mathbf{1}_2$ is the only involution in $SU(2)$, $f^2 = -\mathbf{1}_2$. So $-\mathbf{1}_2 \in \langle f_1, f_2 \rangle$. □

The implication of theorem 4.2 is that each generating set $\{g_1, g_2\}$ of a dihedral or polyhedral group corresponds to four generating sets $\{f_1, f_2\}$, $\{f_1, -f_2\}$, $\{-f_1, f_2\}$, $\{-f_1, -f_2\}$ of the corresponding binary dihedral or binary polyhedral group. If $f \in \Pi^{-1}(\{g\})$ and $\hat{f} \in \Pi^{-1}(\{\hat{g}\})$, then $\hat{g} = h^{-1}gh$ if and only if $\hat{f} = q^{-1}fq$ or $-\hat{f} = q^{-1}fq$. Therefore, given $f_i \in \Pi^{-1}(\{g_i\})$, $\hat{f}_i \in \Pi^{-1}(\{\hat{g}_i\})$ ($i \in \{1, 2\}$), the generating sets $\{g_1, g_2\}$ and $\{\hat{g}_1, \hat{g}_2\}$ are $SO(3)$-equivalent if and only if $\{\hat{f}_1, \hat{f}_2\}$ is $SU(2)$-equivalent to one of the four sets $\{f_1, f_2\}$, $\{f_1, -f_2\}$, $\{-f_1, f_2\}$, and $\{-f_1, -f_2\}$.

In the following, we proceed in the following manner. Firstly we construct $SU(2)$-representations of the cyclic groups C_n and determine all generating sets of these representations. Subsequently we take $SO(3)$-representations of the dihedral and polyhedral groups and find all generating sets up to $SO(3)$-equivalence. By computing the generating sets of the corresponding $SU(2)$-representations, we determine all generating sets of the binary dihedral resp. binary polyhedral groups up to $SU(2)$-equivalence.

Let $\mu, \sigma : F_2 = \langle a, b \rangle \longmapsto G \subseteq SO(3)$ be epimorphisms. An automorphism $\phi : G \longmapsto G$, $\phi(\mu(a)) = \sigma(a)$, $\phi(\mu(b)) = \sigma(b)$ exists if and only if $\mathrm{Ker}(\mu) = \mathrm{Ker}(\sigma)$. Hence, if we know all possibilities of choosing $\mathrm{Ker}(\mu)$, such that $\mathrm{Im}(\mu) = G$, and if we have found all automorphisms of G which are no conjugations in $SO(3)$, then we have determined all generating sets of G up to $SO(3)$-equivalence.

As we mentioned in section 2, $\mathrm{Ker}(\mu)$ can be described by defining relations. But different defining relations can induce the same set $\mathrm{Ker}(\mu)$. Thus, to obtain a general view, we will always characterize $\mathrm{Im}(\mu)$ in the following way:

$$\mathrm{Im}(\mu) := \langle x, y;\ x^n = y^m = (xy)^p = R_4 = \cdots = R_k = \mathbf{1} \rangle \tag{4.1}$$

where $x := \mu(a)$, $y := \mu(b)$; $n, m, p, k \in \mathbb{N}$, $k > 3$.

The next lemma (resp. its proof) is important because it simplifies the computation of generating sets.

Lemma 4.1. Given $\mathrm{Im}(\mu)$ in the manner of (4.1). If $(\hat{n}, \hat{m}, \hat{p})$ is an arbitrary permutation of (n, m, p), then there exists an epimorphism $\rho : F_2 \longmapsto \mathrm{Im}(\mu)$ with $\mathrm{Im}(\mu) = \mathrm{Im}(\rho) = \langle \hat{x}, \hat{y};\ \hat{x}^{\hat{n}} = \hat{y}^{\hat{m}} = (\hat{x}\hat{y})^{\hat{p}} = \hat{R}_4 = \ldots = \hat{R}_k = \mathbf{1} \rangle$, where $\rho(a) = \hat{x}$, $\rho(b) = \hat{y}$. For the definition of the $\hat{R}_j$'s we refer to the proof.

Proof. The set of all permutations of (n, m, p) is S_3, i.e. the symmetric group of rank 3. Hence, to prove the lemma, it is sufficient to verify this statement for the generators of the S_3, e.g. for the transpositions $(1\,2)$ and $(1\,3)$.

Therefore, we have to show the following. If $\mathrm{Im}(\mu) = \langle x, y;\ x^n = y^m = (x\,y)^p = R_4 = \cdots = R_k = \mathbf{1} \rangle$, then there exists τ, ν, so that:

$$\begin{aligned}
\mathrm{Im}(\mu) = \mathrm{Im}(\tau) &:= \langle \hat{x}, \hat{y};\ \hat{x}^m = \hat{y}^n = (\hat{x}\,\hat{y})^p = \hat{R}_4 = \cdots = \hat{R}_k = \mathbf{1} \rangle \\
&\quad \text{with} \quad \tau(a) := \hat{x}, \tau(b) := \hat{y} \\
\mathrm{Im}(\mu) = \mathrm{Im}(\nu) &:= \langle \bar{x}, \bar{y};\ \bar{x}^p = \bar{y}^m = (\bar{x}\,\bar{y})^n = \bar{R}_4 = \cdots = \bar{R}_k = \mathbf{1} \rangle \\
&\quad \text{with} \quad \nu(a) := \bar{x}, \nu(b) := (\bar{y})\,.
\end{aligned}$$

One can verify these statements by using the following definitions:

(i) $\hat{x} := \tau(a) = y^{-1}$; $\hat{y} := \tau(b) = x^{-1}$. $\hat{R}_j$ is defined by substituting in R_j all $\hat{y}^{-1}$ for x and all $\hat{x}^{-1}$ for y.

(ii) $\bar{x} := \nu(a) = x\,y$, $\bar{y} := \nu(b) = y^{-1}$. $\bar{R}_j$ is defined by substituting in R_j all $\bar{x}\,\bar{y}^{-1}$ for x and all $\bar{y}^{-1}$ for y.

Because the substitution rules are invertible, the groups are identical. □

The group of automorphisms Φ_2 of the free group $F_2 = \langle a, b \rangle$ contains elements c_2, c_3 (cf [5]) which are defined by

$$c_2 \begin{cases} a \longmapsto a\,b \\ b \longmapsto b^{-1} \end{cases}$$

$$c_3 \begin{cases} a \longmapsto b^{-1} \\ b \longmapsto a^{-1}\,. \end{cases}$$

Thus the substitutions which are used in the proof of lemma 4.1 can be interpreted as the mapping $(\mu(a), \mu(b)) \longmapsto (\mu(c_j(a)), \mu(c_j(b)))$ $(j \in \{1, 2\})$.

4.2. The generators of C_n

A cyclic group can be characterized by the property that it can be generated by a single element, i.e. $C_n := \langle h; h^n = id \rangle$. In this section we investigate the possibilities of generating C_n by two elements. Given the elements $x, y \in C_n$ with $O(x) = p$, $O(y) = q$ ($p, q \in \mathbb{N}$). The orders of the two elements can be written in the form $p = r\,s$, $q = r\,t$, where $r = \gcd(p, q)$, $\gcd(s, t) = 1$. Because the orders of all elements of C_n must be divisors of $\mathrm{Ord}(C_n)$, n fulfils the equation $n = m\,r\,s\,t$ ($m \in \mathbb{N}$). $\langle x^s \rangle$ as well as $\langle y^t \rangle$ are subgroups of C_n of order r. But, on the other hand, C_n has precisely one subgroup of order r; this is $\langle h^{mst} \rangle$ (cf [7, Chapter II]). Hence, $\langle x^s \rangle = \langle h^{mst} \rangle = \langle y^t \rangle$. This results in $\langle x^s \rangle \subseteq \langle x \rangle \cap \langle y \rangle$, which implies $r \leqslant \mathrm{Ord}(\langle x \rangle \cap \langle y \rangle)$. Because $\mathrm{Ord}(\langle x \rangle \cap \langle y \rangle)$ has to be a divisor of both $\mathrm{Ord}(\langle x \rangle) = rs$ and $\mathrm{Ord}(\langle y \rangle) = rt$, it follows that $\mathrm{Ord}(\langle x \rangle \cap \langle y \rangle) = r$. The product $\langle x \rangle \langle y \rangle := \{a\,b; a \in \langle x \rangle, b \in \langle y \rangle\}$ equals $\langle x, y \rangle$, since C_n is an Abelian group. Therefore

$$\mathrm{Ord}(\langle x, y \rangle) = \mathrm{Ord}(\langle x \rangle \langle y \rangle) = \frac{\mathrm{Ord}(\langle x \rangle)\,\mathrm{Ord}(\langle y \rangle)}{\mathrm{Ord}(\langle x \rangle \cap \langle y \rangle)} = r\,s\,t\,. \tag{4.2}$$

Thus the order of C_n is $n = r\,s\,t$.

Let a faithful representation of C_n be defined by

$$C_n := \left\{ \begin{pmatrix} \mathrm{e}^{2\pi i m/n} & 0 \\ 0 & \mathrm{e}^{-2\pi i m/n} \end{pmatrix}; m \in \{0, 1, \ldots, n-1\} \right\}.$$

As a consequence of the above considerations, all generating sets of the representation of C_n can be written in the form

$$C_n := \left\langle \begin{pmatrix} \mathrm{e}^{2\pi i p/rs} & 0 \\ 0 & \mathrm{e}^{-2\pi i p/rs} \end{pmatrix}, \begin{pmatrix} \mathrm{e}^{2\pi i q/rt} & 0 \\ 0 & \mathrm{e}^{-2\pi i q/rt} \end{pmatrix} \right\rangle \tag{4.3}$$

where $n = r\,s\,t$, $\gcd(p, rs) = \gcd(q, rt) = \gcd(s, t) = 1$.

4.3. The generators of the binary dihedral and the binary polyhedral groups

According to [4, Chapter I, section 19] one can define the finite, noncyclic subgroups of $SO(3)$ by

$$\begin{aligned} D_n &:= \langle x, y;\, x^2 = y^2 = (x\,y)^n = id \rangle \\ T &:= \langle x, y;\, x^3 = y^2 = (x\,y)^3 = id \rangle \\ Y &:= \langle x, y;\, x^5 = y^2 = (x\,y)^3 = id \rangle \end{aligned} \tag{4.4}$$

where we have used the fact that $T \cong A_4$ and $Y \cong A_5$ (A_k the alternating group of rank k).

In order to prove that

$$O \cong S_4 := \langle x, y; x^4 = y^2 = (x\,y)^3 = id \rangle \tag{4.5}$$

(S_4 is the symmetric group of rank 4) let us start from the definitions $r_1 := y$, $r_2 := x^{-1} y\, x$, $r_3 := x^{-2} yx^2$. One can show by using the defining relations of $x^4 = y^2 = (x\,y)^3 = id$, that $(r_1\, r_2)^3 = (r_2\, r_3)^3 = (r_1\, r_3)^2 = id$. For the proof that $S_4 := \langle r_1, r_2, r_3\,; r_1^2 = r_2^2 = r_3^2 = (r_1\, r_2)^3 = (r_2\, r_3)^3 = (r_1\, r_3)^2 = id \rangle$ we refer to [4, chapter I, Bsp 19.7]. Thus, $S_4 \subseteq \langle x, y \rangle$. On the other hand, the relation $\langle x, y \rangle \subseteq S_4$ is valid, since $x = r_3\, r_2\, r_1$, $y = r_1$. Hence, $S_4 := \langle x, y; x^4 = y^2 = (x\,y)^3 = id \rangle$.

In [14] one can find a determination of all possibilities of choosing $\mathrm{Ker}(\mu)$ so that $\mathrm{Im}(\mu)$ is a finite non-cyclic subgroup of $SO(3)$ by a combinatorical approach. For the results we refer to the appendix.

$SO(3)$-representations corresponding to the defining relations (4.4) resp. (4.5) are given by ($\alpha := 2\pi\, p/n$, $\gcd(p,n) = 1$, $\tau := (1+\sqrt{5})/2$)

$$
\begin{aligned}
D_n &:= \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \\
T &:= \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \\
O &:= \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \\
Y &:= \left\langle \frac{1}{2} \begin{pmatrix} \tau & \tau^{-1} & 1 \\ \tau^{-1} & 1 & -\tau^{-1} \\ -1 & \tau & \tau^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle .
\end{aligned}
\tag{4.6}
$$

In the case of D_n the $SO(3)$-representations with defining relations (4.4) which are not $O(3)$-equivalent correspond to different choices of p in (4.6). Each automorphism of T and O can be representated by a conjugation with an element of O. Thus, (4.6) determines each $SO(3)$-representation of T and O with defining relations (4.4) resp. (4.5) up to $SO(3)$-equivalence.

The case of Y is a little bit more complicated. Since the group of automorphisms of A_5 is S_5, there exists a second generating set of A_5, which corresponds to the defining relations $x^5 = y^2 = (x\,y)^3 = id$, and which is not $SO(3)$-equivalent to (4.6); this is

$$
Y := \left\langle \frac{1}{2} \begin{pmatrix} 1 & \tau & \tau^{-1} \\ \tau & -\tau^{-1} & -1 \\ -\tau^{-1} & 1 & -\tau \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & -\tau & \tau^{-1} \\ \tau & -\tau^{-1} & 1 \\ -\tau^{-1} & 1 & \tau \end{pmatrix} \right\rangle . \tag{4.7}
$$

Generating sets of the corresponding $SU(2)$-representations of the binary dihedral resp. binary polyhedral groups are given by ($\beta := \pi\, q/n$, $\gcd(q,n) = 1$, $\tau := 1 + \sqrt{5}/2$)

$$
\begin{aligned}
D_n^* &:= \left\langle \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i}\,e^{-\mathrm{i}\beta} \\ \mathrm{i}\,e^{\mathrm{i}\beta} & 0 \end{pmatrix} \right\rangle \\
T^* &:= \left\langle \frac{1}{2} \begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\rangle \\
O^* &:= \left\langle \frac{1}{2} \begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \right\rangle \\
Y^* &:= \left\langle \frac{1}{2} \begin{pmatrix} \tau & \tau^{-1}+\mathrm{i} \\ -\tau^{-1}+\mathrm{i} & \tau \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} \right\rangle \\
Y^* &:= \left\langle \frac{1}{2} \begin{pmatrix} \tau^{-1} & -1+\mathrm{i}\tau \\ 1+\mathrm{i}\tau & \tau^{-1} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+\mathrm{i}\tau & -\tau^{-1} \\ \tau^{-1} & 1-\mathrm{i}\tau \end{pmatrix} \right\rangle .
\end{aligned}
\tag{4.8}
$$

In order to obtain generating sets of the binary dihedral resp. binary polyhedral groups which correspond to other defining relations one can use the corresponding substitution rules, because Π is a homomorphism. Proceeding in this manner, it remains only to write down the different possibilities of choosing the signs of the matrices, then we have determined all generating sets of the binary dihedral resp. binary polyhedral groups up to $SU(2)$-equivalence.

5. Fibonacci orbits on finite subgroups of $SU(2)$

5.1. General remarks

The elements $M_0 = \mu(a)$ and $M_1 = \mu(b)$ together with recurrence relation (3.2) determine a Fibonacci orbit completely. Two Fibonacci orbits $\{M_n, n \in \mathbb{N}_0\}$ and $\{\hat{M}_n, n \in \mathbb{N}_0\}$ are called $SU(2)$-equivalent, if there exists $g \in SU(2)$, s.t. $g^{-1} M_n g = \hat{M}_n$ for each $n \in \mathbb{N}_0$. Particularly, the Fibonacci orbits are $SU(2)$-equivalent if and only if the generating sets $\{M_0, M_1\}$ and $\{\hat{M}_0, \hat{M}_1\}$ are $SU(2)$-equivalent. Thus, because we have determined all generating sets of finite subgroups of $SU(2)$ up to $SU(2)$-equivalence, the determination of all Fibonacci orbits on finite subgroups up to $SU(2)$-equivalence is reduced to simple matrix recursions. As we mentioned in section 4, the Fibonacci orbits on finite groups are always periodic. Hence, according to definition 3.2, carrying out the matrix recurrence until $M_p = M_0$ and $M_{p+1} = M_1$, we know the entire Fibonacci orbit.

Because the matrix recurrence (3.2) is invertible (cf equation (3.3)), each set $\{M_j, M_{j+1}\}$ is a generating set of the group $\langle M_0, M_1 \rangle$. Thereby, it is easy to check to which equivalence class a generating set belongs. Sutherland [13] has shown, that the generating sets $\{X, Y\}$ and $\{\hat{X}, \hat{Y}\}$ are $SU(2,\mathbb{C})$-equivalent if and only if the equations $\mathrm{tr}(X) = \mathrm{tr}(\hat{X})$, $\mathrm{tr}(Y) = \mathrm{tr}(\hat{Y})$, and $\mathrm{tr}(X\,Y) = \mathrm{tr}(\hat{X}\,\hat{Y})$ are valid. Therefore, the equivalence class of generating sets can be determined by the calculation of the traces.

We have shown in section 3, that odd periods require $K(M_0, M_1) = \pm\mathbf{1}_2$. It follows from $K(M_0, M_1) = \mathbf{1}_2$, that $\langle M_0, M_1 \rangle$ is a cyclic group. Assume that $K(M_0, M_1) = -\mathbf{1}_2$, then $\Pi(\langle M_0, M_1 \rangle)$ must be an Abelian subgroup of $SO(3)$ but $\langle M_0, M_1 \rangle$ has to be a Nonabelian subgroup of $SU(2)$. As a consequence, $\langle M_0, M_1 \rangle$ must be the binary dihedral group D_2^*. Therefore, odd periods are only possible if $\langle M_0, M_1 \rangle$ is a cyclic group or the binary dihedral group D_2^*.

The case that $\langle M_0, M_1 \rangle$ is a cyclic group, is treated in the next subsection. For results relating to Fibonacci orbits on the noncyclic finite subgroups of $SU(2)$ we refer to the appendix.

5.2. Fibonacci orbits on C_n

As we have shown in section 4.2, each generating set of C_n is $SU(2)$-equivalent to a generating set of the form

$$C_n = \left\langle \begin{pmatrix} e^{i2\pi j/rs} & 0 \\ 0 & e^{-i2\pi j/rs} \end{pmatrix}, \begin{pmatrix} e^{i2\pi k/rt} & 0 \\ 0 & e^{-i2\pi k/rt} \end{pmatrix} \right\rangle$$

with $n = r\,s\,t$, $\gcd(s,t) = \gcd(j, rs) = \gcd(k, rt) = 1$. Using the (generalized) Fibonacci numbers, which are defined by $f_0^\ell = 0$, $f_1^\ell = 1$ und $f_{n+1}^\ell = f_{n-1}^\ell + \ell\, f_n^\ell$ $(n \in \mathbb{N})$, we can write M_p as

$$M_p = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix} \qquad (p \in \mathbb{N}) \tag{5.1}$$

where $\gamma = (2\pi/r)((j/s) f_{p-1}^\ell + (k/t) f_p^\ell)$. Thus, the condition of definition 3.2 that a Fibonacci orbit has period p can be written as a system of linear equations:

$$\begin{pmatrix} f_{p-1}^\ell - 1 & f_p^\ell \\ f_p^\ell & f_{p+1}^\ell - 1 \end{pmatrix} \begin{pmatrix} j\,t \\ k\,s \end{pmatrix} \overset{\mathrm{mod}(n)}{\equiv} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.2}$$

This system of linear equations can be solved by calculating the Fibonacci numbers and checking which combinations of j, k, r, s, and t satisfy (5.2). On the other hand, by using the fact that (5.2) is an eigenvalue problem in the commutative ring $\mathbb{Z}/n\mathbb{Z}$, one can find some restrictions on period p. Because

$$\det\begin{pmatrix} f^{\ell}_{p-1} & f^{\ell}_{p} \\ f^{\ell}_{p} & f^{\ell}_{p+1} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & l \end{pmatrix}^{p} = (-1)^{p} \tag{5.3}$$

we obtain the following condition:

$$0 \overset{\mathrm{mod}(n)}{\equiv} \det\begin{pmatrix} f^{\ell}_{p-1}-1 & f^{\ell}_{p} \\ f^{\ell}_{p} & f^{\ell}_{p+1}-1 \end{pmatrix} = 1-(f^{\ell}_{p-1}+f^{\ell}_{p+1})+(-1)^{p}\,. \tag{5.4}$$

Regarding the assumption $\gcd(j,s) = \gcd(k,t) = 1$, equation (5.2) together with the recurrence relation of the Fibonacci numbers results in $f^{\ell}_{p-1} = a\,s\,t$, $f^{\ell}_{p+1} = b\,s\,t$, where $a, b \in \mathbb{Z}$. Thus, the characteristic polynomial (5.3) can be taken in the form

$$(-1)^{p}-1 \overset{\mathrm{mod}(n)}{\equiv} (a+b)\,st\,. \tag{5.5}$$

It follows from this equation that odd periods imply $s\,t = 1$ or $s\,t = 2$, i.e. odd periods are only possible if either both generators are of order n or if one generator is of order n and the other is of order $n/2$ (which implies that n is even).

Another restriction follows from the investigation of the case $n = s$, $r = t = 1$, that is:

$$C_n = \left\langle \begin{pmatrix} e^{i2\pi j/n} & 0 \\ 0 & e^{-i2\pi j/n} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle. \tag{5.6}$$

One can check by a simple computation that in this case $f^{\ell}_{p-1}-1 \overset{\mathrm{mod}(n)}{\equiv} 0$, $f^{\ell}_{p} \overset{\mathrm{mod}(n)}{\equiv} 0$, and $f^{\ell}_{p+1}-1 \overset{\mathrm{mod}(n)}{\equiv} 0$. Consequently, Equation (5.2) is satisfied for each choice of j, t, k, r, s and t, where $n = rst$.

Therefore, in order to determine the period of the Fibonacci orbits, it is reasonable to start from the case (5.6), where the period p must be even for $n \neq 2$. Subsequently, the investigation of periods can be reduced to the divisors of p.

6. The quasiperiodic spin echo

This section deals with an interpretation of the well known spin-echo experiment as periodic dynamics on a binary dihedral resp. a dihedral group. Furthermore we describe a quasiperiodic version of the spin echo experiment.

Let $\vec{B}$ be a time-independent, homogeneous magnetic field in direction of the x_3-axis and a radio-frequency electromagnetic field of frequency $\omega/2\pi$ circularly polarized about an axis in the $x_1\,x_2$-plane. Then the Hamiltonian of a spin-$\frac{1}{2}$-particle has the form

$$H(t) = -\frac{ge\hbar}{4mc}\begin{pmatrix} B_3 & (B_1 - iB_2)\,e^{-i\omega t} \\ (B_1 + iB_2)\,e^{i\omega t} & B_3 \end{pmatrix}. \tag{6.1}$$

where $B_i \in \mathbb{R}$. According to [3, p 158], we obtain the following formula for the time evolution operator under resonance conditions:

$$U(t) = \begin{pmatrix} \cos(\alpha\,t)\,e^{-i\omega t/2} & -i\sin(\alpha\,t)\,e^{i(\omega t/2+\beta)} \\ -i\sin(\alpha\,t)\,e^{i(\omega t/2)+\beta} & \cos(\alpha\,t)\,e^{i\omega t/2} \end{pmatrix} \tag{6.2}$$

with $\omega = -(ge/2mc)B_3$, $\alpha = -(ge/4mc)|B_1 + iB_2|$, $e^{i\beta} = (B_1 + iB_2)/|B_1 + iB_2|$. The expectation of the magnetic moment is given by

$$\vec{\Sigma}(t) = (\Sigma_1(t), \Sigma_2(t), \Sigma_3(t)) \tag{6.3}$$

where $\Sigma_k(t) = \langle\vec{\Psi}(t), \sigma_k \vec{\Psi}(t)\rangle = \langle\vec{\Psi}(0), U^{-1}(t)\, \sigma_k\, U(t)\, \vec{\Psi}(0)\rangle$. In the sequel, we investigate some special choices of $\vec{B}$.

Let us consider the case of free precessing in which the radio-frequency field is switched off, i.e. $B_1 = B_2 = 0$. Then

$$U(t) = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \tag{6.4}$$

which implies

$$\begin{pmatrix} \Sigma_1(t) \\ \Sigma_2(t) \\ \Sigma_3(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_1(0) \\ \Sigma_2(0) \\ \Sigma_3(0) \end{pmatrix}. \tag{6.5}$$

Now, looking at the case $\alpha t = \pi/2 \Leftrightarrow t = -2\pi\, mc/|B_1 + iB_2|\, ge$ in (6.2), we obtain the time evolution matrix

$$U(t) = \begin{pmatrix} 0 & -ie^{-i\rho/2} \\ -ie^{i\rho/2} & 0 \end{pmatrix} \tag{6.6}$$

($\rho = (\pi/2)\, B_3/|B_1 + iB_2| + \beta$). This case is called π-pulse because the time evolution of the expectation

$$\begin{pmatrix} \Sigma_1(t) \\ \Sigma_2(t) \\ \Sigma_3(t) \end{pmatrix} = \begin{pmatrix} \cos\rho & \sin\rho & 0 \\ \sin\rho & -\cos\rho & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Sigma_1(0) \\ \Sigma_2(0) \\ \Sigma_3(0) \end{pmatrix} \tag{6.7}$$

can be interpreted as a rotation through an angle of 180° about an axis in the $x_1\, x_2$-plane. *Note that the matrix of (6.4) together with the matrix of (6.6) is a generating set of a binary* dihedral group. Since the expectations Σ_k are linear functionals which are transformed in the same way as the σ_k's under the action of the $SU(2)$-matrices, the transformation matrices of $\vec{\Sigma}$ are the same as would obtained by using the homomorphism Π, which is described in section 4.1. Thus, the matrices of (6.5) and (6.7) are a generating set of a dihedral group.

The $\pi/2$-pulse is defined analogously to the π-pulse, i.e. $\alpha t = \pi/4$. The time evolution of the expectations is given by

$$\begin{pmatrix} \Sigma_1(t) \\ \Sigma_2(t) \\ \Sigma_3(t) \end{pmatrix} = \begin{pmatrix} \cos 2\lambda & -\sin 2\lambda & 0 \\ 0 & 0 & -1 \\ \sin 2\lambda & \cos 2\lambda & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1(0) \\ \Sigma_2(0) \\ \Sigma_3(t) \end{pmatrix} \tag{6.8}$$

with $\lambda = (\pi/4)\, B_3/|B_1 + B_2| + \beta$.

The spin echo experiment is carried out in the following way: take protons in water and switch on a homogeneous magnetic field in the direction of the x_3-axis. If $kT \ll -geB_3/4mc$ then in the thermodynamic equilibrium the expectation Σ_3 is nonzero while Σ_1 and Σ_2 are zero. Because of the multitude of protons we can identify expectations *and relative frequencies, i.e. the magnetization $\vec{M}$ of the system consisting of N protons is* given by $\vec{M} = N\vec{\Sigma}$.

Now, let us carry out a $\frac{\pi}{2}$-pulse. Assume that the $\frac{\pi}{2}$-pulse ends at time $t = 0$. Then, according to (6.8), the magnetization is non-zero in the $x_1\, x_2$-plane and zero in direction of the x_3-axis. In the periodic spin echo experiment the system passes alternatingly through two types of intervalls, called b and a. In the interval b the protons precess freely while the interval a is a π-pulse.

We start at time $t = 0$ with an interval of type b. Because of the inhomogeneities of the magnetic field in the water there exists no definite angular velocity but a spectrum of velocities, i.e. after the interval b we obtain the magnetization:

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(b) = \int_{-\infty}^{\infty} f(\omega) \begin{pmatrix} \cos(\omega b) & \sin(\omega b) & 0 \\ -\sin(\omega b) & \cos(\omega b) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{d}\omega \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(0) \qquad (6.9)$$

where f is a distribution. To illustrate this time evolution we choose $M_\pm = M_1 \pm \mathrm{i}M_2$:

$$M_+(b) = \int_{-\infty}^{\infty} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega b}\,\mathrm{d}\omega\, M_+(0)$$
$$M_-(b) = \int_{-\infty}^{\infty} f(\omega)\,\mathrm{e}^{\mathrm{i}\omega b}\,\mathrm{d}\omega\, M_-(0)\,.$$

Thus, M_+ and M_- have the same time evolution as wave packets in quantum mechanics: they disperse. This implies that $M_1(b)$ and $M_2(b)$ are zero, too. $M_3(b)$ is zero because $M_3(0)$ is already zero.

Subsequently, carrying out a π-pulse, the magnetization after the interval $b \circ a$ is

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(b \circ a) = \int_{-\infty}^{\infty} f(\omega) \begin{pmatrix} \cos(\omega b) & \sin(\omega b) & 0 \\ \sin(\omega b) & -\cos(\omega b) & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathrm{d}\omega \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(0)\,.$$

We have chosen $\rho = 0$, which corresponds to a special choice of the direction of the field.

Now, let the particles precess freely again. Assume that the Larmor frequency of each particle is the same as in the beginning, then the magnetization after $b \circ a \circ b$ is given by

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(b \circ a \circ b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}(0)\,. \qquad (6.10)$$

Therefore, after the sequence $b \circ a \circ b$ we can measure a macroscopic magnetization again. As a consequence, the equation $\vec{M}(b \circ a \circ b \circ a) = \vec{M}(0)$ is valid. The time evolution in the interval $b \circ a \circ b \circ a$ is shown in figure 1.

Note that, because $aba = b^{-1}$, the spin echo experiment is a realization of a system with reversing symmetry (cf [11]).

A quasiperiodic version of this experiment can be executed in the following manner. Let the intervals I_0 and I_1 be given by $I_0 := a$ and $I_1 := b$, where a and b are defined as mentioned above. Applying the recurrence relation of the Fibonacci sequence ($I_{n+1} = I_n \circ I_{n-1}$ ($n \in \mathbb{N}$)) we can build the Fibonacci chain. For instance, the intervals I_6 and I_7 have the form:

$$\begin{aligned} I_6 &= babbababbabba \\ I_7 &= babbababbabbababb\,abab\,. \end{aligned} \qquad (6.11)$$

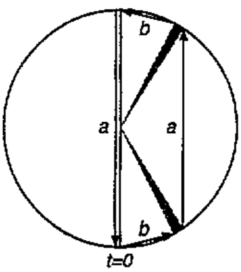

Figure 1. We have chosen the x_1-axis in horizontal and the x_2-axis in the vertical direction.

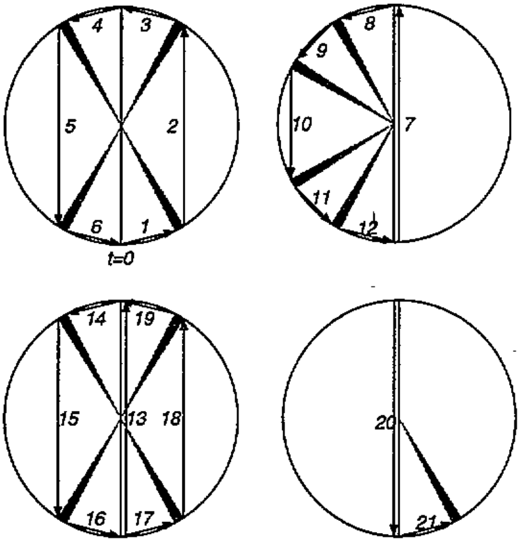

Figure 2. The sequence starts at $t = 0$. The numbers of the arrows show the succession of the intervals.

Starting with a $\frac{\pi}{2}$-pulse, we obtain again a magnetization in the $x_1 x_2$-plane. Let this pulse end at $t = 0$. Reading I_7 from the left to the right, we obtain a time evolution of the magnetization which is illustrated in figure 2.

One can easily check that the period of the Fibonacci orbit is 6, since $\vec{M}(I_6) = \vec{M}(I_0)$ and $\vec{M}(I_7) = \vec{M}(I_1)$.

So far, we have described the protons as isolated particles. But in reality there are proton–proton and proton–water interactions. These interactions result in a relaxation process, which implies a decreasing magnitude of the magnetization in the $x_1 x_2$-plane. A mathematical description of this problem as well as the details of the experiment can be found in Abragam [1].

Another problem of a measurement is that the Fibonacci orbits of the binary dihedral groups correspond to the hyperbolic orbit $(0, y, 0, 0, -y, 0)$ of the Fibonacci trace map (cf [11]). As a consequence, the Fibonacci orbit is unstable, too. Thus, if the measurement is possible at all, we have to measure by sectors, e.g. we measure the first six intervals and then start again with a $\frac{\pi}{2}$-pulse.

7. Concluding remarks

In this paper we started from an interpretation of the Fibonacci sequence as the set of all positive powers of an element of the group of automorphisms of the free group F_2 applied to a generator of F_2. The Fibonacci sequence was mapped by homomorphisms into finite subgroups of $SU(2)$, where the images of the Fibonacci sequences are called Fibonacci orbits. By introducing an equivalence relation between generating sets resp. Fibonacci orbits we reduced the determination of all Fibonacci orbits to the statement of a representative of each equivalence class and a simple matrix recurrence.

As an application we described a two-level system which possesses a quasiperiodic structure. The quasiperiodic structure was a Fibonacci sequence built of two types of time intervals. So, the time evolution of the system became a quasiperiodic $SU(2)$-dynamics.

Acknowledgment

It is a great pleasure to thank M Baake for helpful discussions.

Appendix A.

The Appendix is arranged in the following way. In section A.1 we give the substitution rules which one needs to go from the generating sets in section 4.3 to generating sets of the same non-cyclic finite subgroups of $SO(3)$ with other defining relations. The content of the other sections of the appendix is the characterization of Fibonacci orbits in the binary dihedral and binary polyhedral subgroups of $SU(2)$.

Appendix A.1. The generating sets of finite non-cyclic subgroups

In this section we proceed in the following manner. For each finite non-cyclic subgroup of $SO(3)$ we call $\{x, y\}$ the generating set which fulfils the defining relations of section 4.3. Starting from these generating sets we gave all possibilities of choosing generating sets which correspond to other defining relations up to such transformations which are described in lemma 4.1. Particularly, the substitution rules are given.

In the case of the dihedral groups each generating set fulfils the defining relations $x^2 = y^2 = (xy)^n$ up to the permutations described in lemma 4.1. So, we can restrict ourselves to the binary polyhedral groups.

(i) The tetrahedral group T:

$$T := \langle x_1, y_1; (x_1)^3 = (y_1)^3 = (x_1 y_1)^3 = (x_1^2 y_1)^2 = id \rangle \quad \text{with} \quad x_1 := xy \quad y_1 := x$$

(ii) The octahedral group O:

$$O := \langle x_1, y_1; (x_1)^4 = (y_1)^4 = (x_1 y_1)^3 = (x_1^2 y_1)^2 = id \rangle \quad \text{with} \quad x_1 := x \quad y_1 := x^{-1} y .$$

(iii) The icosahedral group Y:

Table A1.

Defining relations	Substitution rules
$\langle \hat{x}, \hat{y}; \hat{x}^3 = \hat{y}^3 = (\hat{x}\hat{y})^5 = (\hat{y}\hat{x}\hat{y}\hat{x}\hat{y})^2 = id \rangle$	$\hat{x} = y^{-1}x^3, \hat{y} = y, x = \hat{y}\hat{x}\hat{y}\hat{x}$
$\langle \hat{x}, \hat{y}; \hat{x}^5 = \hat{y}^5 = (\hat{x}\hat{y})^2 = (\hat{x}^3\hat{y})^3 = id \rangle$	$\hat{x} = x^2, \hat{y} = x^{-1}y, x = \hat{x}^3, y = \hat{x}^3\hat{y}$
$\langle \hat{x}, \hat{y}; \hat{x}^5 = \hat{y}^5 = (\hat{x}\hat{y})^3 = (\hat{x}^2\hat{y})^2 = id \rangle$	$\hat{x} = x, \hat{y} = x^{-1}y, y = \hat{x}\hat{y}$
$\langle \hat{x}\hat{y}; \hat{x}^5 = \hat{y}^5 = (\hat{x}\hat{y})^3 = (\hat{x}^2\hat{y}^2)^2 = id \rangle$	$\hat{x} = x, \hat{y} = yx^3y^{-1}, y = \hat{x}\hat{y}^2$
$\langle \hat{x}, \hat{y}; \hat{x}^5 = \hat{y}^5 = (\hat{x}\hat{y})^5 = (\hat{x}^2\hat{y})^3 = (\hat{x}^3\hat{y})^2 = id \rangle$	$\hat{x} = x, \hat{y} = x^{-2}y, y = \hat{x}^2\hat{y}$

Appendix A.2. The orbits on D_n^*

Suppose that one fixes the variable q under the constraint $\gcd(q, n) = 1$; then D_n^* has three generating sets up to $SU(2)$-equivalence. Firstly, we regard the cases in which the Fibonacci automorphism is of the form $\rho_{2\ell+1}$ with ($\ell \in \mathbb{Z}$). Because in these cases the period is always 3 or 6, the following table characterizes the Fibonacci orbits completely.

Particularly, given a suitable q, each Fibonacci orbit is the following one up to $SU(2)$-equivalence and up to the choice of the starting point. To facilitate the determination of the equivalence classes of the generating sets we state the traces $x := \frac{1}{2}\mathrm{tr}(M_n)$, $y := \frac{1}{2}\mathrm{tr}(M_{n+1})$, and $z := \frac{1}{2}\mathrm{tr}(M_{n+1}\, M_n)$ ($E(x) := \exp(\mathrm{i}\frac{\pi}{n}x)$).

Table A2.

n	M_n	(x, y, z)
0	$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$	$(0, 0, -\cos\pi q/n)$
1	$\begin{pmatrix} 0 & \mathrm{i}E(q) \\ \mathrm{i}E(-q) & 0 \end{pmatrix}$	$(0, (-1)^{l+1}\cos\pi q/n, 0)$
2	$(-1)^{\ell+1}\begin{pmatrix} E(q) & 0 \\ 0 & E(-q) \end{pmatrix}$	$((-1)^{\ell+1}\cos\pi q/n, 0, 0)$
3	$(-1)^{\ell+1}\begin{pmatrix} 0 & \mathrm{i}E(q(2\ell+2)) \\ \mathrm{i}E(-q(2\ell+2)) & 0 \end{pmatrix}$	$(0, 0, \cos\pi q/n)$
4	$(-1)^{\ell}\begin{pmatrix} 0 & \mathrm{i}E(q(2\ell+1)) \\ \mathrm{i}E(-q(2\ell+1)) & 0 \end{pmatrix}$	$(0, (-1)^{\ell}\cos\pi q/n, 0)$
5	$(-1)^{\ell}\begin{pmatrix} E(-q) & 0 \\ 0 & E(q) \end{pmatrix}$	$((-1)^{\ell}\cos\pi q/n, 0, 0)$

Note that the Fibonacci orbit on the infinite binary dihedral group D^*_∞ has period 6, too.

Provided that the Fibonacci automorphism is $\rho_{2\ell}$ ($\ell \in \mathbb{Z}$) and that we have chosen a suitable q, then there exist two Fibonacci orbits up to $SU(2)$-equivalence and up to the choice of the starting point:

(i)

Table A3.

n	M_n	(x, y, z)
0	$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$	$(0, 0, -\cos\pi q/n)$
1	$\begin{pmatrix} 0 & \mathrm{i}E(q) \\ \mathrm{i}E(-q) & 0 \end{pmatrix}$	$(0, 0, (-1)^{\ell+1}\cos\pi q/n)$
2	$(-1)^{\ell}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$	$(0, 0, -\cos\pi q/n)$
3	$(-1)^{\ell}\begin{pmatrix} 0 & \mathrm{i}E(q) \\ \mathrm{i}E(-q) & 0 \end{pmatrix}$	$(0, 0, (-1)^{\ell+1}\cos\pi q/n)$

In this case the period is 4.

(ii)

$$M_n = \begin{cases} (-1)^{\ell m}\begin{pmatrix} E(q) & 0 \\ 0 & E(-q) \end{pmatrix} & \text{if } n = 2m \\ \begin{pmatrix} 0 & \mathrm{i}E(2mq\ell) \\ \mathrm{i}E(-2mq\ell) & 0 \end{pmatrix} & \text{if } n = 2m+1\,. \end{cases}$$

The period p of these Fibonacci orbits is

$$p = \begin{cases} \dfrac{2n}{ggT(\ell, n)} & \text{if } n \text{ even} \\ \dfrac{4n}{ggT(\ell, n)} & \text{if } n \text{ odd} . \end{cases}$$

If we proceeded with the binary polyhedral groups T^*, O^* and Y^* in the same fashion as we did in the case of D_n^*, we would produce an amount of data which would be confusing rather than instructive. Thus, we confine ourselves to the Fibonacci orbits which belong to the automorphism ρ_1. Additionally, for the sake of brerity, the information about each Fibonacci orbit up to $SU(2)$-equivalence and up to the starting point is reduced to a possible starting point and the period of the Fibonacci orbit.

Appendix A.3. The orbits on T^*

(i) Starting point:

$$M_0 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} .$$

Period: 48

(ii) Starting point:

$$M_0 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} .$$

Period: 16

Appendix A.4. The orbits on O^*

(i) Starting point:

$$M_0 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} \qquad M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} .$$

Period: 18

(ii) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} \qquad M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} .$$

Period: 18

Appendix A.5. The orbits on Y^*

(i) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} \tau & -\tau^{-1}+\mathrm{i} \\ \tau^{-1}+\mathrm{i} & \tau \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} .$$

Period: 50

(ii) Starting point:

$$M_0 = \frac{1}{2}\begin{pmatrix} \tau & -\tau^{-1}+\mathrm{i} \\ \tau^{-1}+\mathrm{i} & \tau \end{pmatrix} \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} .$$

Period: 150

(iii) Starting point:

$$M_0 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i}\tau^{-1} & \mathrm{i}\tau \\ \mathrm{i}\tau & 1-\mathrm{i}\tau^{-1} \end{pmatrix} \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}.$$

Period: 14

(iv) Starting point:

$$M_0 = \frac{1}{2}\begin{pmatrix} 1-\mathrm{i}\tau^{-1} & -\mathrm{i}\tau \\ -\mathrm{i}\tau & 1+\mathrm{i}\tau \end{pmatrix} \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}.$$

Period: 14

(v) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} 1-\mathrm{i}\tau^{-1} & -\mathrm{i}\tau \\ -\mathrm{i}\tau & 1+\mathrm{i}\tau \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}.$$

Period: 42

(vi) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i}\tau^{-1} & \mathrm{i}\tau \\ \mathrm{i}\tau & 1-\mathrm{i}\tau \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}.$$

Period: 42

(vii) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} 1-\mathrm{i}\tau^{-1} & -\mathrm{i}\tau \\ -\mathrm{i}\tau & 1+\mathrm{i}\tau \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} -\tau^{-1}+\mathrm{i}\tau & \mathrm{i} \\ \mathrm{i} & -\tau^{-1}-\mathrm{i}\tau \end{pmatrix}.$$

Period: 4

(viii) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} 1-\mathrm{i}\tau^{-1} & -\mathrm{i}\tau \\ -\mathrm{i}\tau & 1+\mathrm{i}\tau \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} -\tau^{-1}+\mathrm{i}\tau & \mathrm{i} \\ \mathrm{i} & -\tau^{-1}-\mathrm{i}\tau \end{pmatrix}$$

Period: 12

(ix) Starting point:

$$M_0 = \frac{1}{2}\begin{pmatrix} \tau^{-1} & -1+\mathrm{i}\tau \\ 1+\mathrm{i}\tau & \tau^{-1} \end{pmatrix} \qquad M_1 = -\frac{1}{2}\begin{pmatrix} 1+\mathrm{i}\tau & -\tau^{-1} \\ \tau^{-1} & 1-\mathrm{i}\tau \end{pmatrix}.$$

Period: 50

(x) Starting point:

$$M_0 = -\frac{1}{2}\begin{pmatrix} \tau^{-1} & -1+\mathrm{i}\tau \\ 1+\mathrm{i}\tau & \tau^{-1} \end{pmatrix} \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i}\tau & -\tau^{-1} \\ \tau^{-1} & 1-\mathrm{i}\tau \end{pmatrix}.$$

Period: 150

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